Last time: the Discrete Fourier Transform

Idea: Consider the vector space

$$V = \{ \psi : G \to \mathbb{C} \}$$

where $G = \mathbb{Z}/N$

V has an orthonormal basis

$$\{ |x\rangle \mid x \in G \}$$

satisfying $\langle x | \psi \rangle = \psi(x)$

$\forall x \in G, \ \psi \in V.$

This is called the position basis, and its elements are called position eigenstates

Last time we showed that there is another orthonormal basis

$$\left\{ \frac{1}{\sqrt{N}} |\chi\rangle \;\middle|\; \chi : G \to U(1) \text{ is a group homomorphism} \right\}$$

which satisfies $\langle x | \frac{1}{\sqrt{N}} | \chi \rangle$

$$= \frac{1}{\sqrt{N}} \chi(x)$$

This is the momentum basis

and its elements are called

momentum eigenstates

<u>Def:</u> The set $\hat{G} \doteq \{ \chi : G \to U(1) \mid \chi \text{ is a (continuous) group hom.} \}$

We classified elements of $\hat{G}$ using Nth

roots of unity.

We can index the N such group homomorphisms

by $k \in \{0, \ldots, N-1\}$

The kth homomorphism is

$$\chi_k : G \to U(1)$$
$$x \mapsto \exp\left(\frac{2\pi i}{N} k x\right)$$

We usually write $|k\rangle$ for $\frac{1}{\sqrt{N}} |\chi_k\rangle$.

We defined the Discrete Fourier Transform of a function $\psi : G \to \mathbb{C}$ to be the function $\tilde{\psi} : \{0, \dots, N-1\} \to \mathbb{C}$

$$k \mapsto \langle k | \psi \rangle$$

(so $\tilde{\psi}(k) = \langle k | \psi \rangle$)

We computed

$$\tilde{\psi}(k) = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} \exp\left(\frac{-2\pi i}{N} k x\right) \psi(x).$$

To do physics, we introduce operators which are diagonal in the position and momentum bases, respectively.

For $G = \mathbb{Z}/N$, we have a

- Position operator

$$\hat{X} \doteq \sum_{x=0}^{N-1} \sin\left(\frac{2\pi}{N} x\right) |x\rangle\langle x|$$

- Momentum operator

$$\hat{P} \doteq \sum_{k=0}^{N-1} \sin\left(\frac{2\pi}{N} k\right) |k\rangle\langle k|$$

Remark: If $\Delta\psi(x) = \frac{\psi(x+1) - \psi(x-1)}{2}$ then $\hat{P} = -i\Delta$

Time for quantum physics!

The setup:

We have a quantum system V that is allowed to evolve over time.

What does it do?

Let a point in time be represented by $t \in \mathbb{R}$.

The Schrödinger picture says that, given $t_1, t_2 \in \mathbb{R}$, that there is a unitary operator $U(t_2, t_1)$, called the time evolution operator, which performs the instruction

Let time evolve from $t = t_1$ to $t = t_2$

Furthermore, these operators satisfy

- $U(t,t) = I$
- $U(t_3, t_2)\, U(t_2, t_1) = U(t_3, t_1)$
- $U : \mathbb{R} \times \mathbb{R} \to \mathrm{End}(V)$ is a smooth function.

Write $U(t) = U(t, 0)$

Then $\mathcal{U}(t_2, t_1) = \mathcal{U}(t_2)\, \mathcal{U}(t_1)^{-1}$

since $= \mathcal{U}(t_2, 0)\, \mathcal{U}(0, t_1)$.

We may see later that these properties imply that there is a self-adjoint operator $\hat{H}$, called the Hamiltonian operator of the system, for which it holds that

$$i \frac{d}{dt} \mathcal{U}(t) = \hat{H}\, \mathcal{U}(t)$$

Schrödinger's eqn

An eigenstate of $\hat{H}$ is called an energy eigenstate

The (useful form of) Schrödinger's equation is an equational identity constraining $\hat{H}$ in terms of $\hat{X}$ and $\hat{P}$ (possibly depending on time)

Most commonly, this equation looks like

$$\hat{H} = \frac{1}{2}\hat{P}^2 + V(\hat{X}),$$

where V is some function of $\hat{X}$.

We'll often want to have complicated functions for V. How do evaluate these on $\hat{X}$?

If $V: G \to \mathbb{R}$ is any function, then we define $V(\hat{X}) = \sum_{x=0}^{N-1} V(x)\,|x\rangle\langle x|$.

The function V is called the potential function of the system

If we are in the position eigenstate $|x\rangle$, (i.e. the current state is $c|x\rangle$)

then we say the potential energy of the current state is $V(x)$

Similarly, if we are in a momentum eigenstate $|k\rangle$, then the kinetic energy of the current state is $\langle k|\frac{1}{2}\hat{P}^2|k\rangle$

$$= \frac{1}{2}\sin\left(\frac{2\pi}{N}k\right)^2.$$

If we measure the system relative to the position eigenbasis, we call it "Measuring the position"
and if we measure relative to the momentum eigenbasis, we call it "measuring the momentum".

In general, given any normal operator

$\hat{T}$, measuring relative to its eigenspace decomposition is called "measuring $\hat{T}$".

If $\hat{T}$ is self-adjoint, then all its eigenvalues are real, and we call such operators the observables of the system.

Ex: $G = \mathbb{Z}/21$ $\quad V(x) = \begin{cases} -20 & \text{if } x \in \{-2,-1,0,1,2\} \\ 0 & \text{Otherwise} \end{cases}$

see Mathematica file.

Now we want to solve

$$i \frac{d}{dt} U(t) = \hat{H} U(t)$$

equiv, $\frac{d}{dt} U(t) = -i \hat{H} \, U(t)$

We now make the simplifying assumption

that $\hat{H}$ does not depend on time.
(time-independent Schrödinger equation).

If $\hat{H}$ were a number, then we would have
an equation of the form $\frac{d}{dt} f(t) = c f(t)$
$\uparrow$ constant
and we know how to solve this
$$\rightarrow \quad f(t) = Ae^{ct} \quad \text{for some constant } A$$

In fact, we can solve Schrödinger's
equation the same way.
$$U(t) = \exp(-it\hat{H}).$$
But what does the exponential of an operator mean?
As above, we can define this in terms
of a diagonalization. But there is a more general way:

Def: Given an operator $\hat{L}$ on a normed space

V, we define the exponential of $\hat{L}$ to be $\exp(\hat{L}) = \sum_{n=0}^{\infty} \frac{(\hat{L})^n}{n!}$

when this series converges. ☆ in what topology?

To define convergence of a sequence of operators, we need a topology on the space of operators.

Def: An operator L on a normed space V is said to be bounded if there exists $C > 0$ such that

$$\|Lv\| \leq C\|v\| \quad \forall v \in V.$$

Problem $\|v_1 e_1 + \dots + v_n e_n\| = (|v_1| + \dots + |v_n|)$

This is only true for the 1-norm defined in the last problem of the warm-up. (But its sufficient to check just this case!)

Given a normed space V, the set of all bounded operators on V is denoted $B(V)$.

Thm: • $B(V)$ is a vector space
• The composition of two bounded operators is also bounded.

In fact we can make $B(V)$ into a normed space.

Def: The function from $B(V)$ to $\mathbb{R}$ denoted $L \longmapsto \|L\|_\infty = \|L\|$ and defined by

$$\|L\| = \inf\{C > 0 \mid \|Lv\| \le C\|v\| \text{ for all } v \in V\}.$$

is called the operator norm or the sup norm.

Thm: The sup norm is a norm.

A common topology to use for $B(V)$ is the norm topology.

("weak-star")

(Another common choice is the weak-* topology. We may discuss this later.)